

On Hypermomentum in General Relativity

III. Coupling Hypermomentum to Geometry

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(Z. Naturforsch. 31 a, 823–827 [1976]; received May 12, 1976)

In Part I¹ of this series we presented the notion of the material hypermomentum current and motivated its introduction into general relativity. In Part II² we showed that a general, linearly connected manifold with symmetric metric (L_4, g) is the appropriate geometrical framework for such an introduction. The present paper completes the picture by giving dynamical definitions for energy-momentum and hypermomentum for a minimally coupled material Lagrangian. We derive and discuss the field equations of a new metric-affine gravitational theory which embodies these notions.

1. Dynamical Definition of Hypermomentum

We now propose the dynamical definition of hypermomentum as the variation of the material Lagrangian density with respect to the affine connection in the space (L_4, g) . That is, we set **

$$e \Delta_k^{ji} := - \frac{\delta \mathcal{L}}{\delta \Gamma_{ij}^k}, \quad (1)$$

where the minimally coupled material Lagrangian density $\mathcal{L}(\psi, \nabla\psi)$ depends on the metric g_{ij} and the connection Γ_{ij}^k as independent variables. We postulate that the tensor Δ^{kji} so defined is in fact the hypermomentum tensor described in Part I.

Using the decomposition (II.3) of the connection, the definition (1) is easily shown to be equivalent to the separate relations for spin

$$\tau^{kji} := \Delta^{[kji]} = e^{-1} \frac{\delta \mathcal{L}}{\delta M_{ijk}} \quad (2)$$

defined by varying $\mathcal{L}(\psi, \partial\psi, g, \partial g, M, Q_i, \bar{Q})$ with respect to the (generalized) contortion tensor, for the dilatation current

$$\Delta^i := \Delta_i^{ji} = -2 e^{-1} \delta \mathcal{L} / \delta Q_i \quad (3)$$

obtained by varying with respect to the Weyl vector, and for the traceless proper hypermomentum

$$\bar{\Delta}^{kji} := \Delta^{(kji)} - \frac{1}{4} g^{kj} \Delta^i = -2 e^{-1} \delta \mathcal{L} / \delta \bar{Q}_{ijk} \quad (4)$$

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** Symbols not defined here are defined in Parts I and II or in Hehl, von der Heyde, Kerlick, and Nester³. We have $e := (-\det g_{ij})^{1/2}$ and $k := 8\pi G/c^4$ where G is Newton's gravitational constant.

obtained by varying with respect to the traceless nonmetricity tensor. In this way we have exploited the analogy between Eqs. (I.8) and (II.5).

One can show by means of the Rosenfeld identities satisfied by the material Lagrangian that the hypermomentum dynamically defined by Eq. (1) is indeed the same quantity that arises from the canonical definition

$$e \Delta_k^{ji} \equiv - \frac{\partial \mathcal{L}}{\partial (\partial_i \psi)} f_k^j \psi, \quad (5)$$

where f_k^j are the representations of the generators of general coordinate transformations. The anti-symmetric part and the trace with respect to j and k of Eq. (5) are the well-known canonical definitions of the spin angular momentum current and the (intrinsic) dilatation current in special relativity, respectively. In anholonomic coordinates, the quantities f_k^j represent the generators of the general linear group $GL(4, R)$ of which the Lorentz group and the dilatations are subgroups.

This last remark seems to preclude the definition of a canonical hypermomentum tensor for spinor fields, since $GL(4, R)$ has no spinor representations. For the time being, at least, we exclude spinor fields from our consideration if we deal with geometries more general than a Y_4 .

The definition (1) and its corollaries (2, 3, 4) provide a consistent procedure for linking the notion hypermomentum, enunciated and given a physical interpretation in Part I, with the geometry (L_4, g) proposed in Part II. Our remaining task is the construction of a consistent gravitational theory which incorporates these notions and accords with our experience. We shall thus require that the new theory which results will, like the U_4 theory, reduce



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to Einstein's theory in the realm of macroscopic physics.

Recall that the metric condition $Q_{ijk} = 0$ is postulated from the very beginning in the U_4 theory. We now propose to drop this postulate and to see to what extent it can be derived (at least in some limit) from the unconstrained metricaffine theory.

2. Field Equations

We now complete this dualistic field theory by introducing an action function for the gravitational field expressed in geometrical variables. Since we would like to compare the theory which results with the U_4 theory and with general relativity, we choose, instead of the independent variables (g_{ij}, Γ_{ij}^k) , the equivalent set of independent quantities $(g_{ij}, S_{ij}^{\cdot k}, Q_{ijk})$. As Fig. 1 of Part II clearly shows, it is the vanishing of the nonmetricity Q_{ijk} which characterizes a U_4 and the vanishing of torsion $S_{ij}^{\cdot k}$ together with the vanishing of Q_{ijk} which characterizes the Riemannian spacetime V_4 of general relativity. We take as the gravitational field Lagrangian density a scalar density \mathcal{V} which depends on g, S, Q , and their derivatives. The variational principle for the interacting system of matter and gravitational field is then

$$\delta \int \left[\frac{1}{2k} \mathcal{V}(g, S, Q) + \mathcal{L}(\psi, \nabla\psi, g) \right] d^4x = 0. \quad (6)$$

The gravitational field equations are the Euler-Lagrange equations of this variation with respect to the geometrical variables,

$$-\frac{1}{ke} \frac{\delta \mathcal{V}}{\delta g_{ij}} = \sigma^{ij} := \frac{2}{e} \frac{\delta \mathcal{L}(g, S, Q)}{\delta g_{ij}}, \quad (7)$$

$$-\frac{1}{2ke} \frac{\delta \mathcal{V}}{\delta S_{ij}^{\cdot k}} = \mu_k^{ji} := \frac{1}{e} \frac{\delta \mathcal{L}(g, S, Q)}{\delta S_{ij}^{\cdot k}}, \quad (8)$$

$$\frac{1}{ke} \frac{\delta \mathcal{V}}{\delta Q_{ijk}} = \nu^{kji} := -\frac{2}{e} \frac{\delta \mathcal{L}(g, S, Q)}{\delta Q_{ijk}}. \quad (9)$$

The first field eq. (7) results from variation with respect to the metric tensor and is a generalization of Einstein's equation of general relativity. The source for the metric is the (L_4, g) generalization of Hilbert's metric energy-momentum tensor σ^{ij} . The second field eq. (8) resembles the second field equation of the U_4 theory which relates torsion to the spin energy potential tensor μ_k^{ji} , here also generalized to an (L_4, g) . The third equation is completely new, making its appearance for the first time

in an (L_4, g) and primarily determining the non-metricity in terms of a new source ν^{kji} . It is easy to rewrite the second and third field equations in the more suggestive, unified form

$$\frac{1}{2e} \frac{\delta \mathcal{V}}{\delta \Gamma_{ij}^k} = k \Delta_k^{ji}, \quad (10)$$

where

$$\Delta^{kji} = \nu^{kji} - \mu^{kji}. \quad (11)$$

Up to this point we have left the gravitational field Lagrangian \mathcal{V} undetermined; its choice is the subject of the next section.

3. Gravitational Field Lagrangian

The obvious first guess for a gravitational field Lagrangian is the curvature scalar in an (L_4, g) since the U_4 theory and Einstein's theory result from this choice in the cases of U_4 and V_4 geometries, respectively. The corresponding density \mathcal{R} can be reduced to an effective first-order Lagrangian and varied with respect to metric and connection as computed in ³. The field equations which result are the pair

$$G^{(ij)} = k {}^r \sigma^{ij} := \frac{2k}{e} \frac{\delta \mathcal{L}(g, \Gamma)}{\delta g_{ij}}, \quad (12)$$

$$\frac{1}{2e} \frac{\delta \mathcal{R}}{\delta \Gamma_{ij}^k} = : P_k^{ji} = k \Delta_k^{ji}, \quad (13)$$

where G^{ij} is the Einstein tensor of an (L_4, g) , ${}^r \sigma^{ij}$ differs from σ^{ij} by a divergence, and P_k^{ji} is a geometrical tensor computed in Ref. ³ whose properties are noted in the Appendix of this article.

Particularly troublesome among the properties of P_k^{ji} is that its trace P_k^{ki} vanishes identically due to the projective invariance of the curvature scalar. The vanishing of P_k^{ki} implies via Eq. (13) the inconsistent relation

$$\Delta^i{}_i \stackrel{?}{=} 0, \quad (14)$$

inconsistent because the dilatation current, a physical current which does not vanish in general, is here required to vanish. The projective invariance of \mathcal{R} also means that the Weyl vector Q_i is left undetermined.

The symmetric traceless part of Eq. (13)

$$P_{(kj)}^{\cdot i} = k \bar{\Delta}_{kj}^{\cdot i} \quad (15)$$

is mathematically consistent and can be resolved to give \bar{Q} in terms of $\bar{\Delta}$:

$$\bar{Q}_{ijk} = k[-2(\bar{\Delta}_{ijk} - \bar{\Delta}_{jki} + \bar{\Delta}_{kij}) + g_{jk}\bar{\Delta}_{ii}^j], \quad (16)$$

but the antisymmetric part

$$P_{[kj]}^{\dots i} = k \tau_{kj}^{\dots i} \quad (17)$$

contains terms which depend on Q_i , so that it is not possible to determine the torsion uniquely in terms of the spin. Similarly, the Einstein-like Eq. (12) for the metric contains Q_i -dependent terms.

We note in passing that the projective invariance of \mathcal{R} and the inconsistent eq. (14) already cause trouble in a Y_4 theory with scalar curvature Lagrangian. The other field equations of that theory are obtainable by setting $\bar{\Delta} = 0$ in Eqs. (12) and (17).

The projective invariance of \mathcal{R} has been known since Weyl's time (see for example Schouten⁴). Trautman⁵ (see also Kopczyński⁶) encountered this difficulty in his derivation of the U_4 theory. He proved that one must assume, in addition to the vanishing of what we call $\Delta^{(kj)i}$, the vanishing of the Weyl vector Q_i . This can be seen from the more detailed examination of $P_{kj}^{\dots i}$ in the Appendix, see in particular Equation (26). Trautman did not, however, provide any physical interpretation for the quantity $\Delta^{(kj)i}$ or argue for its existence.

Sandberg⁷ has proposed the modification of the matter Lagrangian in an (L_4, g) so that it too is projectively invariant, but this procedure seems physically unjustified to us. We rather expect that the difficulty will be resolved by replacing the gravitational field Lagrangian by an expression quadratic in the curvature tensor or by adding quadratic terms to \mathcal{R} .

We would propose as a general guideline for modifying the field Lagrangian that the theory which results should reduce to the U_4 theory upon the vanishing of $\Delta^{(kj)i}$ and to Einstein's theory when τ^{kji} is also set equal to zero. An ad hoc modification, in line with these principles but otherwise physically unmotivated, is to add a term $(e/2\alpha)Q_i Q^i$ to \mathcal{R} . In such a model theory, Eqs. (12), (15), and (17) retain their general form, but the "bad" eq. (14) is replaced by the "good" equation $Q^i = \alpha k \Delta^i$.

Another possible escape from this difficulty might be a "dynamical breaking" of the projective symmetry of the field Lagrangian if this could be arranged in a physically reasonable way.

4. The Dilatation Current in a Y_4 and Elementary Particle Physics

In Part I we mentioned that the dilatation current is conserved exactly in the high-energy "scaling limit" of elementary particle physics. This current J^k contains both an intrinsic part Δ^k whose canonical definition is the trace of Eq. (5) and an orbital part constructed from the canonical energy momentum tensor Σ^{lk} . Thus, in special relativity,

$$J^k = \Delta^k + x_l \Sigma^{lk}. \quad (18)$$

Note that in the (L_4, g) framework the symmetric part of Σ^{lk} obeys the relation $\Sigma^{(lk)} = \Gamma \sigma^{lk}$. The special relativistic conservation law for J^k is $\nabla_k J^k = \Theta_k^k$, where Θ_k^k is the "soft trace" of the "improved" energy-momentum tensor that goes to zero in the scaling limit (compare Callan, Coleman, and Jackiw⁸). Hence the conservation law for Δ^k is

$$\nabla_k \Delta^k = \Theta_k^k - \Sigma_k^k. \quad (19)$$

A form similar to (19) can be obtained from the identities satisfied by the material Lagrangian in a Y_4 . [Recall from Eq. (3) that a Y_4 is the appropriate framework for incorporating the dilatation current via a dynamical definition.] In a Y_4 with a minimally coupled material Lagrangian, we find from taking the trace of the canonical and metric energy momentum tensors that the conservation law for Δ^k can be expressed as

$$\frac{1}{4} g_{ij}^* \nabla_k (e g^{ij} \Delta^k) = e (\tilde{\Theta}_k^k - \Sigma_k^k), \quad (20)$$

where $\nabla_k^* := \nabla_k + 2 S_{kl}^{\dots i}$ reduces to a divergence on vector densities. Here we have set

$$e \tilde{\Theta}^{ij} = e \sigma^{ij} + 2 \nabla_k^* (e \tau^{k(ij)}). \quad (21)$$

Since a gravitational theory in a Y_4 should arise from local gauge invariance with respect to the Weyl group (the Poincaré group plus dilatations, eleven parameters in all), we look for this procedure to suggest a satisfactory Y_4 field Lagrangian. Local gauge theories for the Weyl group are derived and discussed in Bregman⁹ and Charap and Tait¹⁰, for instance.

5. Proper Hypermomentum in General Relativity, Conclusions *

We have found a consistent means for linking up the concept of hypermomentum with the most general metricaffine geometry (L_4, g) . The dynamical

definitions which provide that link are both internally consistent and physically suggestive.

Unfortunately, the simplest generalization of the field Lagrangian of general relativity to an (L_4, g) runs into trouble because of its projective invariance. It is easy enough to construct a new Lagrangian which is not beset by these difficulties and which has the correct limits, but a natural choice for this new Lagrangian has yet to be found.

A gravitational theory based on an (L_4, g) and embodying the canonical and dynamical definitions of hypermomentum should arise as the local gauge theory of the general affine group $GA(4, R)$ over spacetime with an additional local Minkowski structure. The general affine group is the semidirect product of the linear group $GL(4, R)$ with the translations and includes the Weyl group as a subgroup. The translational gauge potentials in such a local gauge theory will be the (4×4) tetrad components and the $GL(4, R)$ potentials will be the (4×16) connection coefficients in anholonomic coordinates. The formalism used by the authors in Ref. ³ can be extended to this more general group by enlarging the set of group generators and providing them with physical interpretations.

We have been led to the new concept of hypermomentum by analogies with continuum mechanics and by arguments from geometry and classical field theory rather than by the known symmetries of elementary particles. The notion of the general affine group, like that of the Poincaré group, originates from the concept of local frames of vectors and their transformations rather than from anything in microphysics. Thus we may well ask whether we can extend these concepts to the microphysical domain. Certainly invariance with respect to the Poincaré group is well established in elementary particle physics by experiment, and dilatation invariance in the high-energy scaling limit seems to be well fulfilled. The other nine elements of $GA(4, R)$ which generate shearlike distortions do not seem to lead to any observed exact symmetries or conserved currents in particle physics (though perhaps it is fair to say that such symmetries and currents have not

been sought). However, a proper hypermomentum need not to be a conserved current in order to be taken seriously. It is certainly possible to construct a canonical proper hypermomentum current for a massive vector field, and, if such a field turns out to be elementary (i. e. not reducible into more elementary spinor fields), this can be taken as an argument in support of the hypermomentum hypothesis.

In Part II we sounded a cautionary note about the causal properties of an (L_4, g) connection in which \bar{Q} is nonvanishing. We can now make the following comments: If acausal phenomena occur, they will probably be confined within matter, as exemplified by the field eq. (16), or at least within such a short range that quantum mechanical "smearing out" of the light cone can still be expected. Therefore we expect the motion of test particles outside matter to be perfectly normal.

Whether proper hypermomentum exists is an open question subject in principle to experimental verification. Independent of this question, the formalism developed in this series of notes has already been a valuable aid to ordering our knowledge of the relation between affine and metric structures in general relativity.

Appendix

The geometrical Tensor P_k^{ji}

Explicit computation of the variation of the (L_4, g) curvature scalar density with respect to the connection yields ³

$$P_{ij}^{..k} = T_{ij}^{..k} - \delta_{[i}^k Q_{j]} + \delta_{[i}^k \bar{Q}_{j]}^{..l} \quad (22)$$

where $T_{ij}^{..k} := S_{ij}^{..k} + 2 \delta_{[i}^k S_{j]}^{..l}$ is the modified torsion tensor, $Q_i := Q_i^{..l}/4$ the Weyl vector, and $\bar{Q}_{ij}^{..k}$ the traceless nonmetricity tensor. The symmetric and antisymmetric parts of (22) with respect to the indices i and j are given by

$$P_{(ij)}^{..k} = -\frac{1}{2} \bar{Q}_{(ij)}^{..k} + \frac{1}{2} \delta_{(i}^k \bar{Q}_{j)}^{..l}, \quad (23)$$

$$\begin{aligned} P_{[ij]}^{..k} &= T_{ij}^{..k} - \delta_{[i}^k Q_{j]} - \frac{1}{2} \bar{Q}_{[ij]}^{..k} + \frac{1}{2} \delta_{[i}^k \bar{Q}_{j]}^{..l} \\ &= -M_{[ij]}^{..k} - \delta_{[i}^k M_{j]}^{..l} + \delta_{[i}^k M_{j]}^{..l}. \end{aligned} \quad (24 a, b)$$

The trace P_k^{ki} vanishes identically, so the symmetric part (23) can be expressed entirely in terms of the traceless nonmetricity tensor \bar{Q} . From Eq. (24 a) the U_4 limit $P_{[ij]}^{..k} = T_{ij}^{..k}$ can be easily read off; then, in Eq. (24 b), the (generalized) contortion $M_{ij}^{..k}$ degenerates to $K_{ij}^{..k}$. It is now a straightforward matter to compute the relations between

* In Part I the following misprints should be corrected: Page 112, 1st column, 3rd line from below, read " $\bar{\Delta}^{\alpha\beta\gamma}$ " (instead of $\bar{\Delta}^{\alpha\beta}$); page 113, line before Eq. (6), insert "laws" after conservation. Erratum: Page 113, 2nd column, first two lines, read "... the divergence relation that the total dilatation current is conserved only if the trace of Θ_{ij} vanishes, that is in the ...".

the symmetries of P_{ijk} and the geometry of the (L_4, g) manifold as given in Table 1 (see also Schrödinger¹¹ and Trautman⁵).

$P_{[ij]k}=0 \Leftrightarrow \Gamma_{ij}^k = \left\{ \begin{smallmatrix} k \\ i \ j \end{smallmatrix} \right\} + \frac{1}{2} Q_{ij}^{\cdot k}$	(L_4, g) with vanishing contortion	(25)
$P_{(ij)k}=0 \Leftrightarrow Q_{ijk}=Q_{i g j k}$	Y_4	(26)
$P_{ijk}=0 \Leftrightarrow \Gamma_{ij}^k = \left\{ \begin{smallmatrix} k \\ i \ j \end{smallmatrix} \right\} + \frac{1}{2} Q_i \delta_j^k$	Y_4 with vanishing contortion or $V_4(*\Gamma)$	(27)

Table 1. Symmetries of P_{ijk} and spacetime geometries.

The starred affinity $*\Gamma$ in (27) is projectively invariant and defined by

$$*\Gamma_{ij}^k := \Gamma_{ij}^k - \frac{2}{3} \delta_j^k S_{il}^{\cdot l}. \quad (28)$$

Note that for the connection in (27) $S_{ij}^{\cdot k} = \frac{1}{2} Q_{[i} \delta_{j]}^k$. Therefore either $Q_i = 0$ or $S_{ik}^{\cdot k} = 0$ leads to the Christoffel connection of a V_4 .

We would like to acknowledge helpful discussions with Prof. P. Mittelstaedt. One of us (G.D.K.) expresses his thanks to the Humboldt Foundation for the award of a fellowship.

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